THE NÉRON COMPONENT SERIES OF AN ABELIAN VARIETY

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ABSTRACT. We introduce the Néron component series of an abelian variety A over a complete discretely valued field. This is a power series in $\mathbb{Z}[[T]]$, which measures the behaviour of the number of components of the Néron model of A under tame ramification of the base field. If A is tamely ramified, then we prove that the Néron component series is rational. It has a pole at $T = 1$, whose order equals one plus the potential toric rank of A. This result is a crucial ingredient of our proof of the motivic monodromy conjecture for abelian varieties. We expect that it extends to the wildly ramified case; we prove this if A is an elliptic curve, and if A has potential purely multiplicative reduction.

1. INTRODUCTION

Let K be a complete discretely valued field with ring of integers R and separably closed residue field k. We denote by p the characteristic exponent of k. Let A be an abelian variety over K. We denote its Néron model by $\mathcal{A}$, and the special fiber of $\mathcal{A}$ by $\mathcal{A}_s$. The group Φ_A of connected components of $\mathcal{A}_s$ is a finite abelian group, whose order we denote by $\phi(A)$. In this paper, we study how Φ_A and $\phi(A)$ vary under finite extension of the base field K.

We denote by $\mathbb{N}'$ the set of strictly positive integers that are prime to p, and we fix a separable closure K^s of K. For each element d of $\mathbb{N}'$, there exists a unique extension $K(d)$ of K in K^s. We define the *Néron component series* $S_\phi(A;T)$ of A as

$$S_\phi(A;T) = \sum_{d \in \mathbb{N}'} \phi(A \times_K K(d))T^d \in \mathbb{Z}[[T]]$$

We will show in Theorem 6.5 that, when A is tamely ramified, the series $S_\phi(A;T)$ is rational, and we determine the order of its pole at $T = 1$ (it equals one plus the *potential toric rank* of A). Recall that A is tamely ramified iff the minimal extension L of K where A acquires semi-abelian reduction, is a tame extension of K.

This result is of independent interest, but our main motivation lies elsewhere: it is a crucial ingredient of our proof of the *motivic monodromy conjecture* for abelian varieties, a global version of Denef and Loeser's motivic monodromy conjecture for complex hypersurface singularities. Our results on $S_\phi(A;T)$ allow us to prove the rationality of the motivic zeta function $Z_A(\mathbb{L}^{-s})$ of A, and to determine the order of its (unique) pole. We've shown that this pole is equal to Chai's base change

conductor $c(A)$ of A [9]. Our proof of the conjecture will appear in [14], see also [13] for a preliminary version of the paper.

The key technical result in the present paper is Theorem 5.7. We denote by $t(A)$ the reductive rank of $\mathcal{A}_s^o$. Under the assumption that A is tamely ramified, we prove that

$$\phi(A \times_K K') = d^{t(A)}\phi(A)$$

for every finite extension K' of K such that $d = [K' : K]$ is prime to $[L : K]$. The main tool we use in order to establish Theorem 5.7 is the theory of *rigid uniformization*, in the sense of [8].

It is natural to ask what kind of properties $S_\phi(A; T)$ might have if A is *wildly* ramified. We expect that Theorem 6.5 (in particular, the rationality of $S_\phi(A; T)$) should also hold for wildly ramified abelian varieties. We prove this if A has purely multiplicative reduction, and if A is an elliptic curve (Proposition 6.8). The general case will be investigated in future work.

We conclude this introduction with a short overview of the paper. In Section 2, we gather some preliminaries on rigid uniformization and the smooth rigid and formal sites. In Section 3 we prove some auxiliary results on maximal subtori of algebraic groups in the context of Néron models. In Section 4 we discuss semi-abelian and good reduction of semi-abelian varities. These three sections form the technical preparation for the main results of the paper.

Section 5 is devoted to proving the key result Theorem 5.7, making extensive use of the results in [8]. In Section 6, we prove the rationality of the Néron component series, and we compute the order of its pole at $T = 1$ (Theorem 6.5 and Proposition 6.8).

2. Preliminaries

2.1. **Notation.** We denote by R a complete discrete valuation ring, with residue field k and quotient field K. We assume that k is separably closed. We denote by p the characteristic exponent of k, and by $\mathbb{N}'$ the ordered set of strictly positive integers that are prime to p. For every R-scheme X, we denote by $\widehat{X}$ its formal completion along the special fiber.

For every field F, we fix a separable closure F^s. We denote by I the inertia group $Gal(K^s/K)$. For each element d of $\mathbb{N}'$, the field K has a unique degree d extension in K^s, which we denote by $K(d)$. We denote by $R(d)$ the normalization of R in $K(d)$.

We denote by

$$(\cdot)_K : (Schemes/R) \to (Schemes/K)$$

the generic fiber functor, and by

$$(\cdot)_s : (Schemes/R) \to (Schemes/k)$$

the special fiber functor. We denote by $(stft/R)$ the category of separated formal R-schemes locally topologically of finite type, by (Rig/K) the category of rigid K-varieties, and by

$$(\cdot)_\eta : (stft/R) \to (Rig/K)$$

the generic fiber functor.

For every field F, an algebraic F-group is a group F-scheme that is locally of finite type. For every smooth algebraic K-group G that admits a Néron lft-model

$\mathcal{G}$, we denote by Φ_G the constant abelian group

$$\pi_0(\mathcal{G}_s) = \mathcal{G}_s/\mathcal{G}_s^o$$

of connected components of $\mathcal{G}_s$, and by $\phi(G) \in \mathbb{N} \cup \{\infty\}$ the cardinality of Φ_G.

If H is a group scheme over a scheme S, and n a positive integer, then we denote by $_nH$ the kernel of multiplication by n on H. Likewise, if H is a constant group, we denote by $_nH$ the subgroup of elements killed by n.

2.2. Rigid uniformization and smooth topology.

Let A be an abelian K-variety. We denote by L the minimal extension of K in K^s where A acquires semi-abelian reduction, and we put $e = [L : K]$. Recall that A is tamely ramified iff e is prime to p (since the residue field k of K is separably closed).

A *rigid uniformization* of A consists of the following data [8, 1.1]:

- a semi-abelian K-variety E that is the extension of an abelian K-variety B with potential good reduction by a K-torus T:

$$0 \to T \to E \to B \to 0$$

- a lattice M in E, of rank $\dim(T)$, and a faithfully flat morphism of rigid K-varieties

$$E^{an} \to A^{an}$$

  with kernel M^{an}. Here $(\cdot)^{an}$ denotes the rigid analytic GAGA functor.

Beware that the group K-scheme M is not necessarily constant, only locally constant for the étale topology.

Let K' be a finite extension of K in K^s, of degree d. If we denote by $(\cdot)'$ the base change functor $(Sch/K) \to (Sch/K')$, then the data

$$0 \longrightarrow T' \longrightarrow E' \longrightarrow B' \longrightarrow 0$$

$$0 \longrightarrow (M')^{an} \longrightarrow (E')^{an} \longrightarrow (A')^{an} \longrightarrow 0$$

define a rigid uniformization of $A' = A \times_K K'$. The objects M and T split over L, and $B \times_K L$ has good reduction.

We denote by R' the normalization of R in K', and by k' the residue field of R'. The latter is a finite purely inseparable extension of k. We'll consider the small rigid smooth sites $(\mathrm{Sp}\, K)_{sm}$ and $(\mathrm{Sp}\, K')_{sm}$, and the small formal smooth sites $(\mathrm{Spf}\, R)_{sm}$ and $(\mathrm{Spf}\, R')_{sm}$ [8, §4]. We obtain a commutative diagram

$$
\begin{array}{ccc}
(\mathrm{Sp}\, K')_{sm} & \xrightarrow{\ j'\ } & (\mathrm{Spf}\, R')_{sm} \\
{\scriptstyle h_K}\downarrow & & \downarrow{\scriptstyle h} \\
(\mathrm{Sp}\, K)_{sm} & \xrightarrow[\ j\ ]{} & (\mathrm{Spf}\, R)_{sm}
\end{array}
$$

We denote by $(Ab_K), \ldots$ the category of abelian sheaves on $(\mathrm{Sp}\, K)_{sm}, \ldots$ For every commutative algebraic K-group X, we denote by $\mathscr{F}_X$ the abelian sheaf on $(\mathrm{Sp}\, K)_{sm}$ represented by X^{an} [8, 3.3]. Note that $(h_K)^*\mathscr{F}_X$ is canonically isomorphic to $\mathscr{F}_{X \times_K K'}$.

Remark. Beware that the functor

$$X \mapsto \mathscr{F}_X$$

from the category of commutative algebraic K-groups to the category (Ab_K) is *not* faithful. For instance, if K has characteristic $p > 1$ and

$$X = \alpha_p = \operatorname{Spec} K[x]/(x^p)$$

then $\mathscr{F}_X = 0$. $\qquad\square$

If X is a smooth commutative algebraic K-group that admits a Néron lft-model $\mathcal{X}$ over R, then the formal completion $\widehat{\mathcal{X}}$ is a formal Néron model for X^{an} [7, 6.2], and $\widehat{\mathcal{X}}$ represents the sheaf $j_*\mathscr{F}_X$ on $(\operatorname{Spf} R)_{sm}$.

Lemma 2.1. *The functors*

$$
\begin{aligned}
(h_K)^* &: (Ab_K) \to (Ab_{K'}) \\
(h_K)_* &: (Ab_{K'}) \to (Ab_K)
\end{aligned}
$$

are exact.

Proof. Since h_K is smooth, the functor $(h_K)^*$ is simply the restriction from $(\operatorname{Sp} K)_{sm}$ to $(\operatorname{Sp} K')_{sm}$, so it is exact. It remains to show that $(h_K)_*$ is right exact. Let $\mathscr{F} \to \mathscr{G}$ be a surjection of abelian sheaves on $(\operatorname{Sp} K')_{sm}$. Let K'' be a finite Galois extension of K containing K'. The morphism $g : \operatorname{Sp} K'' \to \operatorname{Sp} K$ is a covering in the rigid smooth site on $\operatorname{Sp} K$, so that it suffices to show that

$$\alpha : g^*(h_K)_*\mathscr{F} \to g^*(h_K)_*\mathscr{G}$$

is surjective. If we denote by g' the morphism $\operatorname{Sp} K'' \to \operatorname{Sp} K'$, then $g = h_K \circ g'$ and we have for every abelian sheaf $\mathscr{H}$ on $(\operatorname{Sp} K')_{sm}$ a canonical isomorphism

$$g^*(h_K)_*\mathscr{H} \cong \bigoplus_{\gamma:K'\to K''} (g')^*\mathscr{H}$$

where γ runs over the morphisms of K-algebras $K' \to K''$. Surjectivity of α now follows from right exactness of $(g')^*$. $\qquad\square$

2.3. **The trace map.** We keep the notations of Section 2.2. Let $\mathscr{F}$ be an abelian sheaf on $(\operatorname{Sp} K)_{sm}$, and consider the tautological morphism

$$\tau : \mathscr{F} \to (h_K)_*(h_K)^*\mathscr{F}$$

We will define a *trace map*

$$tr : (h_K)_*(h_K)^*\mathscr{F} \to \mathscr{F}$$

such that the composition $tr \circ \tau$ is multiplication by $d = [K' : K]$.

Let K'' be a Galois extension of K that contains K'. The morphism $g : \operatorname{Sp} K'' \to \operatorname{Sp} K$ is a covering in the rigid smooth site on $\operatorname{Sp} K$, and we have a canonical isomorphism

$$g^*(h_K)_*(h_K)^*\mathscr{F} \cong \bigoplus_{\gamma:K'\to K''} g^*\mathscr{F}$$

where γ runs over the morphisms of K-algebras $K' \to K''$. Consider the morphism

$$tr_{K''} : \bigoplus_{\gamma:K'\to K''} g^*\mathscr{F} \to g^*\mathscr{F} : (\alpha_\gamma) \mapsto \sum_\gamma \alpha_\gamma$$

It is invariant under the action of the Galois group $Gal(K''/K)$. This implies that $tr_{K''}$ satisfies the gluing conditions w.r.t. the covering g, so that $tr_{K''}$ descends to a morphism of sheaves

$$tr : (h_K)_*(h_K)^*\mathscr{F} \to \mathscr{F}$$

on $(\mathrm{Sp}\,K)_{sm}$. The composition $tr \circ \tau$ is multiplication by d, since this holds after base change to K''.

3. Toric rank

3.1. **Subtori of algebraic groups.** We recall some results on subtori of algebraic groups. We focus on the case of smooth commutative algebraic groups over a field.

Definition 3.1. *Let G be an algebraic group over a field F. A subtorus T of G is called a maximal subtorus of G if, for some algebraically closed field F' containing F, the algebraic group $G \times_F F'$ does not admit any subtorus that is strictly larger than $T \times_F F'$.*

If this property holds for some algebraically closed extension F', then it holds for all algebraically closed overfields of F [1, XII.1.2].

Proposition 3.2. *Let G be a smooth commutative algebraic group over a field F.*
  (1) *The algebraic group G admits a unique maximal subtorus.*
  (2) *If T is an F-torus, then every morphism of algebraic F-groups $T \to G$ factors through the maximal subtorus of G.*

Proof. (1) Existence follows from [1, XIV.1.1], uniqueness from [1, XII.7.1(b)] and commutativity of G.

(2) Denote by S the maximal subtorus of G. We have to show that all morphisms of algebraic F-groups
$$f : T \to G/S$$
are trivial. We may assume that F is algebraically closed, and that G is connected. Since S is the maximal subtorus of G, we know by the Chevalley decomposition [10] that G/S is the extension of an abelian variety A by a unipotent F-group U. Since T does not admit any non-trivial morphism to U [1, XVII.2.4] or to A [10, 2.3], we see that f is trivial. □

Definition 3.3. *Let G be a smooth commutative algebraic group over a field F. The reductive rank $\rho(G)$ of G is the dimension of the maximal subtorus of G.*

Proposition 3.4. *Let G be a smooth commutative algebraic group over a field F. If T is the maximal subtorus of G, then*
$$\rho(G/T) = 0$$
and there exist no non-trivial morphisms from an F-torus to G/T.

Proof. By Proposition 3.2, it suffices to show that $\rho(G/T) = 0$. Let S be a subtorus of G/T. Then we have a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & G/T & \longrightarrow & 0 \\
 & & {=}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T & \longrightarrow & G \times_{G/T} S & \longrightarrow & S & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $G \times_{G/T} S$ is an extension of two F-tori, it is again an F-torus (it is of multiplicative type by [1, IX.8.2], and it is smooth and connected). Since T is the

maximal subtorus of G, we have that $T \to G \times_{G/T} S$ is an isomorphism, so that S must be trivial. $\qquad\square$

Proposition 3.5. *If G is a smooth commutative algebraic group over a field F, then G admits a unique split subtorus T_{sp} such that for every split F-torus S and every morphism of algebraic F-groups $f : S \to G$, the morphism f factors through T_{sp}.*

If G is a torus with character module $X(G)$, then the dimension of T_{sp} is equal to the rank of the free $\mathbb{Z}$-module $X(G)^{Gal(F^s/F)}$.

Proof. By Proposition 3.2, we may assume that G is a torus. Let F' be a splitting field of G, denote by $X(G)$ the character module of G, and consider the trace map

$$tr : X(G) \to X(G)^{Gal(F'/F)} : x \mapsto \sum_{\gamma \in Gal(F'/F)} \gamma \cdot x$$

By the duality between tori and their character modules, it is clear that T_{sp} is the torus corresponding to the character module $X(G)/ker(tr)$ (cf. [16, 1.3]). Since the restriction of tr to

$$X(G)^{Gal(F'/F)} = X(G)^{Gal(F^s/F)}$$

is multiplication by $[F' : F]$, we see that

$$\dim(T_{sp}) = \mathrm{rank}_{\mathbb{Z}}(X(G)/ker(tr)) = \mathrm{rank}_{\mathbb{Z}}(X(G)^{Gal(F^s/F)})$$

$\qquad\square$

Definition 3.6. *With the notations of Proposition 3.5, we call T_{sp} the maximal split subtorus of G.*

An argument similar to the proof of Proposition 3.4 shows that the maximal split subtorus of G/T_{sp} is trivial.

3.2. **Toric rank of a semi-abelian variety.**

Definition 3.7. *Let G be a semi-abelian K-variety, with Néron lft-model $\mathcal{G}$. We define the toric rank $t(G)$ of G by*

$$t(G) = \rho(\mathcal{G}_s^o)$$

Lemma 3.8. *Let*

$$f : \mathcal{H} \to \mathcal{G}$$

be a morphism of smooth group R-schemes such that f_K is injective. Then

$$\rho(\mathcal{G}_s^o) \geq \rho(\mathcal{H}_s^o)$$

Proof. Denote by S and T the maximal subtori of $\mathcal{G}_s^o$, resp. $\mathcal{H}_s^o$. We consider the commutative diagram

$$
\begin{array}{ccccc}
{}_\ell T(k) & \xrightarrow{(*)} & {}_\ell \mathcal{H}_s^o(k) & \longrightarrow & {}_\ell \mathcal{G}_s^o(k) \\
& & \uparrow & & \uparrow \\
& & {}_\ell \mathcal{H}^o(R) & \xrightarrow{(*)} & {}_\ell \mathcal{G}^o(R)
\end{array}
$$

The vertical arrows are bijections, by [6, 7.3.3], and the arrows marked by (*) are injections. It follows that

$${}_\ell T(k) \to {}_\ell \mathcal{G}_s^o(k)$$

is injective. The morphism $T \to \mathcal{G}_s^o$ factors through S, by Proposition 3.2(2). Injectivity of

$$_{\ell}T(k) \to {}_{\ell}S(k)$$

implies that $\dim(S) \geq \dim(T)$, since the number of elements of $_{\ell}\mathbb{G}_m^d(k)$ equals ℓ^d, for every integer $d \geq 0$. □

Proposition 3.9. *Let G be a semi-abelian K-variety. For every finite extension K' of K, we have*

$$t(G) \leq t(G \times_K K')$$

with equality if G has semi-abelian reduction.

Proof. We denote by R' the normalization of R in K', by k' its residue field, and by $\mathcal{G}$ and $\mathcal{G}'$ the Néron lft-models of G and G'. We consider the unique R'-morphism

$$\mathcal{G} \times_R R' \to \mathcal{G}'$$

extending the canonical isomorphism between the generic fibers. Applying Lemma 3.8 to this morphism, we see that

$$t(G) \leq t(G \times_K K')$$

If G has semi-abelian reduction, then

$$\mathcal{G}_s^o \times_k k' \to (\mathcal{G}')_s^o$$

is an isomorphism [2, IX.3.1(e)], so that

$$t(G) = t(G \times_K K')$$

□

Definition 3.10. *Let G be a semi-abelian K-variety, and let L be a finite separable extension of K such that $G \times_K L$ has semi-abelian reduction. We define the potential toric rank $t_{pot}(G)$ of G by*

$$t_{pot}(G) = t(G \times_K L)$$

By Proposition 3.9, this definition does not depend on L, and we have

$$t_{pot}(G) = \max\{t(G \times_K K') \,|\, K' \text{ a finite extension of } K\}$$

We say that G has *purely multiplicative reduction* if $t_{pot}(G)$ is equal to the dimension of G, i.e., if the identity component of the special fiber of the Néron model of $G \times_K L$ is a torus.

Remark. The existence of L (i.e., the potential semi-abelian reduction of G) is well-known. It is easily deduced from the semi-abelian reduction theorem for abelian varieties [2, IX.3.6]; see the implication (2) $\Rightarrow$ (1) in Proposition 4.1 below. □

Definition 3.11. *Let G be a semi-abelian K-variety, with toric part G_{tor} and abelian part G_{ab}. We say that G has good reduction if G_{tor} and G_{ab} have good reduction. We say that G has potential good reduction if there exists a finite separable extension K' of K such that $G \times_K K'$ has good reduction.*

Note that every algebraic K-torus has potential good reduction.

Proposition 3.12. *Let G be a semi-abelian K-variety with potential good reduction. If we denote by T_{sp} the maximal split subtorus of G, then*

$$
\begin{array}{rcl}
t(G) & = & \dim(T_{sp}) \\
t_{pot}(G) & = & \rho(G)
\end{array}
$$

Proof. The second equality follows from the first, passing to a splitting field of the maximal subtorus of G. So let us prove the first equality. We denote by $\mathcal{G}$ the Néron lft-model of G.

Case 1: T_{sp} is trivial, and G is a torus. By [3, 2.3.1], $\mathcal{G}^o$ is affine. Consider a morphism

$$f_s : \mathbb{G}_{m,k} \to \mathcal{G}_s^o$$

By [1, IX.7.3], the morphism f_s lifts uniquely to a morphism of group R-schemes $f : \mathbb{G}_{m,R} \to \mathcal{G}^o$. Passing to the generic fiber, we find a morphism of algebraic K-groups $f_K : \mathbb{G}_{m,K} \to G$. Since T_{sp} is a point, f_K, and hence f_s, must be trivial, so that $t(G) = 0$.

Case 2: T_{sp} is trivial. We have to show that $t(G) = 0$. We denote by

$$0 \to G_{tor} \to G \to G_{ab} \to 0$$

the Chevalley decomposition of G, and by

$$\mathcal{G}_{tor}^o \to \mathcal{G}^o \to \mathcal{G}_{ab}^o$$

the induced sequence on identity components of Néron lft-models. Taking formal completions and passing to the generic fiber, we find a sequence of rigid K-groups

$$(\widehat{\mathcal{G}}_{tor}^o)_\eta \to (\widehat{\mathcal{G}}^o)_\eta \to (\widehat{\mathcal{G}}_{ab}^o)_\eta$$

We denote by $\mathbb{G}_{m,K}^{rig}$ the generic fiber of the formal group R-scheme $\widehat{\mathbb{G}}_{m,R}$. It is a rigid K-group, and it coincides with the unit circle in the rigid analytification $(\mathbb{G}_{m,K})^{an}$.

Assume that there exists a non-trivial morphism of algebraic k-groups

$$f_s : \mathbb{G}_{m,k} \to \mathcal{G}_s^o$$

By the infinitesimal lifting property for tori [1, IX.3.6] we know that this morphism lifts uniquely to a morphism of formal group R-schemes $f : \widehat{\mathbb{G}}_{m,R} \to \widehat{\mathcal{G}}^o$. Passing to the generic fiber, we find a morphism of rigid K-groups $f_\eta : \mathbb{G}_{m,K}^{rig} \to (\widehat{\mathcal{G}}^o)_\eta$.

We consider the composed morphism $g_\eta : \mathbb{G}_{m,K}^{rig} \to (\widehat{\mathcal{G}}_{ab}^o)_\eta$. By the universal property of the formal Néron model [7, 1.1], g_η extends uniquely to a morphism of formal group R-schemes $g : \widehat{\mathbb{G}}_{m,R} \to \widehat{\mathcal{G}}_{ab}^o$. Passing to the special fiber, we obtain a morphism of algebraic k-groups $g_s : \mathbb{G}_{m,k} \to (\mathcal{G}_{ab})_s^o$. By Proposition 3.9, the fact that G_{ab} has potential good reduction implies that $t(G_{ab}) = 0$. Hence, g_s is trivial. It follows from [1, IX.3.5] that g is trivial, so that the image of f_η is contained in $(\widehat{\mathcal{G}}_{tor}^o)_\eta$. But $t(G_{tor}) = 0$ by Case 1, so repeating the above argument we see that f_η is trivial, so that f_s is trivial. Hence, $t(G) = 0$.

Case 3: general case. We denote by $\mathcal{T}$ the Néron lft-model of T_{sp}. We consider the unique R-morphism

$$f : \mathcal{T}^o \to \mathcal{G}^o$$

extending $T_{sp} \to G$. By Lemma 3.8, we have

$$t(G) \geq \dim(T_{sp})$$

It remains to prove the converse inequality.

If we put $H = G/T_{sp}$, then $t(H) = 0$ by Case 2. Copying the proof of Case 2, we see that the image of any morphism of rigid K-groups $\mathbb{G}_{m,K}^{rig} \to G^{an}$ is contained in T_{sp}^{an}, and that any morphism of algebraic k-groups $\mathbb{G}_{m,k} \to \mathcal{G}_s^o$ lifts to a morphism $\mathbb{G}_{m,k} \to \mathcal{T}_s^o$. Hence,

$$t(G) \leq \dim(T_{sp})$$

$\square$

Proposition 3.13. *Let A be an abelian K-variety. We adopt the notation of Section 2.2. We consider M as a discrete I-module, and we denote by T_{sp} the maximal split subtorus of T. Then we have*

$$\begin{array}{ccccc} \mathrm{rank}_{\mathbb{Z}}(M) & = & \dim(T) & = & t_{pot}(A) \\ \mathrm{rank}_{\mathbb{Z}}(M^I) & = & \dim(T_{sp}) & = & t(A) \end{array}$$

Proof. Since T and M split over L, the first statement follows from the second. If we denote by $\mathcal{E}$ the Néron lft-model of E, then $\mathcal{E}_s^o$ is isomorphic to $\mathcal{A}_s^o$ [8, 2.3]. It follows from Proposition 3.12 that

$$t(E) = \dim(T_{sp})$$

so we find

$$t(A) = \dim(T_{sp})$$

Since T_{sp} is split, we know that $R^1 j_* \mathscr{F}_{T_{sp}} = 0$ [8, 4.2]. By [8, 4.4+9+11+12] we have exact sequences

$$0 \longrightarrow \Phi_{T_{sp}} \longrightarrow \Phi_E \longrightarrow \Phi_{E/T_{sp}}$$

$$0 \longrightarrow M^I \longrightarrow \Phi_E \longrightarrow \Phi_A$$

But $\Phi_{E/T_{sp}}$ and Φ_A are finite [6, 10.2.1] so that

$$\mathrm{rank}_{\mathbb{Z}}(M^I) = \mathrm{rank}_{\mathbb{Z}}(\Phi_E) = \mathrm{rank}_{\mathbb{Z}}(\Phi_{T_{sp}}) = \dim(T_{sp})$$

where the last equality follows from the description of the Néron lft-model of $\mathbb{G}_{m,K}$ in [6, 10.1.5]. $\square$

4. SEMI-ABELIAN REDUCTION OF SEMI-ABELIAN VARIETIES

Proposition 4.1. *Let G be a semi-abelian K-variety, with toric part G_{tor} and abelian part G_{ab}. The following are equivalent:*

(1) *G has semi-abelian reduction*
(2) *G_{ab} has semi-abelian reduction, and G_{tor} is split*
(3) *the action of I on $T_\ell G$ is unipotent.*

Proof. The sequence of Tate modules

$$0 \to T_\ell G_{tor} \to T_\ell G \to T_\ell G_{ab} \to 0$$

is exact. Points (2) and (3) are equivalent if G_{tor} or G_{ab} is trivial, by [2, IX.3.8] and the canonical isomorphism of I-modules

$$T_\ell(G_{tor}) \cong X(G_{tor})^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

where $X(G_{tor})^\vee$ is the cocharacter module of G_{tor}. It follows that (2) and (3) are equivalent for arbitrary G.

If (2) holds, then by [6, 10.1.7], the sequence of identity components of Néron lft-models

$$0 \to \mathcal{G}^o_{tor} \to \mathcal{G}^o \to \mathcal{G}^o_{ab} \to 0$$

is exact, so that $\mathcal{G}^o_s$ is semi-abelian.

So it suffices to prove the implication $(1) \Rightarrow (2)$. We denote by g, g_{tor} and g_{ab} the dimensions of G, G_{tor} and G_{ab}, respectively.

Assume that G_{tor} is split. Then, again by [6, 10.1.7], the sequence of identity components of Néron lft-models

$$0 \to \mathcal{G}^o_{tor} \to \mathcal{G}^o \to \mathcal{G}^o_{ab} \to 0$$

is exact. Since $(\mathcal{G}_{tor})^o_s$ is a torus and $\mathcal{G}^o_s$ semi-abelian, we see that $(\mathcal{G}_{ab})^o_s$ is semi-abelian.

Hence, it is enough to show that G_{tor} is split, or equivalently, that $t(G_{tor}) = g_{tor}$ (Proposition 3.12). We denote by T_{tor}, T and T_{ab} the maximal subtori of $(\mathcal{G}_{tor})^o_s$, $\mathcal{G}^o_s$ and $(\mathcal{G}_{ab})^o_s$, respectively.

Consider the commutative diagram

$$
\begin{array}{ccccc}
0 \longrightarrow & {}_\ell\mathcal{G}^o_{tor}(R) & \longrightarrow & {}_\ell\mathcal{G}^o(R) & \longrightarrow & {}_\ell\mathcal{G}^o_{ab}(R) \\
& \downarrow & & \downarrow & & \downarrow \\
& {}_\ell(\mathcal{G}_{tor})^o_s(k) & \longrightarrow & {}_\ell\mathcal{G}^o_s(k) & \longrightarrow & {}_\ell(\mathcal{G}_{ab})^o_s(k) \\
& f\uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & {}_\ell T_{tor}(k) & \longrightarrow & {}_\ell T(k) & \longrightarrow & {}_\ell T_{ab}(k)
\end{array}
$$

The first row is exact, the upper vertical arrows are bijections by [6, 7.3.3], and the lower vertical arrows are injective. Moreover, since $(\mathcal{G}_{tor})^o_s$ is affine [3, 2.3.1], we know that $U := (\mathcal{G}_{tor})^o_s/T_{tor}$ is unipotent [1, XVII.7.2.1], so that ${}_\ell U = 0$ and f is bijective. It follows that the third row is exact, too. Looking at the cardinality of its entries, we see that

$$\ell^{\dim(T) - \dim(T_{tor})} \leq \ell^{\dim(T_{ab})}$$

and, hence, that

$$(4.1) \qquad\qquad t(G) - t(G_{tor}) \leq t(G_{ab})$$

Let K' be a finite separable extension of K such that $G_{tor} \times_K K'$ splits, and $G_{ab} \times_K K'$ has semi-abelian reduction. If we denote by $\mathcal{G}'_{tor}$, $\mathcal{G}'$ and $\mathcal{G}'_{ab}$ the Néron models of $G_{tor} \times_K K'$, $G \times_K K'$ and $G_{ab} \times_K K'$, respectively, then the sequence

$$0 \to (\mathcal{G}'_{tor})^o_s \to (\mathcal{G}')^o_s \to (\mathcal{G}'_{ab})^o_s \to 0$$

is exact [6, 10.1.7]. This implies that

$$(4.2) \qquad\qquad t_{pot}(G_{ab}) = t_{pot}(G) - t_{pot}(G_{tor})$$

On the other hand, by (4.1) and Proposition 3.9, we find

$$
\begin{array}{rcl}
t_{pot}(G) - t_{pot}(G_{tor}) & = & t(G) - t_{pot}(G_{tor}) \\
& \leq & t(G) - t(G_{tor}) \\
& \leq & t(G_{ab}) \\
& \leq & t_{pot}(G_{ab})
\end{array}
$$

Since the first and last term of the inequality are equal by (4.2), we may conclude that

$$t(G_{tor}) = t_{pot}(G_{tor}) = g_{tor}$$

<div align="right">□</div>

Corollary 4.2. *Let G be a semi-abelian K-variety with potential good reduction, or an abelian K-variety. Denote by e the degree of the minimal extension L of K where G acquires semi-abelian reduction. If K' is a finite separable extension of K such that $[K' : K]$ is prime to e, then*

$$t(G) = t(G \times_K K')$$

Proof. Assume that G is an abelian K-variety. If E is the semi-abelian variety appearing in the rigid uniformization of G, then E has potential good reduction, and $t(A) = t(E)$ by [8, 2.3]. Hence, it suffices to consider the case where G is a semi-abelian K-variety with potential good reduction.

Denote by T the maximal subtorus of G, and by T_{sp} the maximal split subtorus of T. By Proposition 3.12, it suffices to show that $T'_{sp} := T_{sp} \times_K K'$ is the maximal split subtorus of $T' = T \times_K K'$. By Proposition 3.5, it is enough to show that

$$X(T)^I = X(T)^{I'}$$

where $X(T)$ is the character module of T, and $I' = Gal(K^s/K')$.

We choose an embedding of L in K^s. We know that T splits over L, by Proposition 4.1. Hence, $Gal(K^s/L)$ acts trivially on the character module $X(T)$ of T. Since $[L : K]$ is prime to $[K' : K]$, we know that the restriction morphism

$$Gal(K'L/K') \to Gal(L/K)$$

is an isomorphism, so that

$$X(T)^I = X(T)^{I'}$$

<div align="right">□</div>

Proposition 4.3. *Let G be a semi-abelian K-variety, with toric part G_{tor} and abelian part G_{ab}.*

The following are equivalent:

(1) *G has good reduction*
(2) *G has semi-abelian reduction, and G_{ab} has potential good reduction*
(3) *the action of I on $T_\ell G$ is trivial.*

Moreover, G has potential good reduction iff G_{ab} has potential good reduction.

Proof. It follows immediately from the definition that G has potential good reduction iff G_{ab} has potential good reduction. The sequence of Tate modules

$$0 \to T_\ell G_{tor} \to T_\ell G \to T_\ell G_{ab} \to 0$$

is exact, so that I acts trivially on $T_\ell G_{ab}$ and $T_\ell G_{tor}$ if I acts trivially on $T_\ell G$. Then G_{ab} has good reduction, by the criterion of Néron-Ogg-Shafarevich [2, IX.2.2.9], and G_{tor} is split, by the canonical isomorphism of I-modules

$$T_\ell(G_{tor}) \cong X(G_{tor})^\vee \otimes_\mathbb{Z} \mathbb{Z}_\ell$$

where $X(G_{tor})^\vee$ is the cocharacter module of G_{tor}. This proves (3) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) follows from Proposition 4.1.

It remains to show that $(1) \Rightarrow (3)$. This can be proven in the same way as [2, IX.2.2.9], namely, by noting that the free $\mathbb{Z}_\ell$-modules

$$T_\ell \mathcal{G}_s^o \cong T_\ell \mathcal{G}^o(R) = (T_\ell G)^I$$

and $T_\ell G$ both have rank $2g_{ab} + g_{tor}$, with g_{ab} and g_{tor} the dimensions of G_{ab}, resp. G_{tor}. □

Proposition 4.4. *If G is a semi-abelian K-variety, then there exists a canonical isomorphism*

$$(\Phi_G)_\ell \cong H^1(I, T_\ell G)_{\mathrm{tors}}$$

where $(\Phi_G)_\ell$ denotes the ℓ-primary part of Φ_G (the subgroup of elements killed by a power of ℓ), and $H^1(I, T_\ell G)_{\mathrm{tors}}$ the torsion part of $H^1(I, T_\ell G)$.

Proof. This is a generalization of [2, IX.11.3.8], and the proof remains valid. □

5. Behaviour of the component group under ramification

Lemma 5.1. *Consider a commutative diagram of abelian groups*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \overset{f}{\longrightarrow} & P_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
& & \alpha \downarrow \wr & & \beta \downarrow & & \downarrow \gamma & & \wr \downarrow \delta & & \\
0 & \longrightarrow & M_2 & \longrightarrow & N_2 & \underset{g}{\longrightarrow} & P_2 & \longrightarrow & Q_2 & \longrightarrow & 0
\end{array}
$$

with exact rows, where α and δ are isomorphisms. Then f induces an isomorphism $ker(\beta) \cong ker(\gamma)$, and g induces an isomorphism $coker(\beta) \cong coker(\gamma)$.

Proof. Note that $M_1 \cap ker(\beta) = \{0\}$ and that $M_2 \subset im(\beta)$. Dividing the first row by M_1 and the second by M_2, we may assume that $M_1 = M_2 = 0$. Now the result follows from an easy diagram chase. □

In [8, 4.7], Bosch and Xarles constructed the identity component $\mathcal{F}^o$ for an arbitrary abelian sheaf $\mathcal{F}$ on $(\mathrm{Spf}\, R)_{sm}$. If the sheaf $\mathcal{F}$ is representable by a formal group R-scheme $\mathfrak{X}$, then the identity component $\mathfrak{X}^o$ represents the sheaf $\mathcal{F}^o$. The component sheaf of $\mathcal{F}$ is defined by

$$\Phi_{\mathcal{F}} = \mathcal{F}/\mathcal{F}^o$$

If $\mathcal{X}$ is a smooth commutative group R-scheme and $\mathcal{F}$ is represented by the formal completion $\widehat{\mathcal{X}}$, then $\Phi_{\mathcal{F}}$ is the constant sheaf on $(\mathrm{Spf}\, R)_{sm}$ associated to the group $\pi_0(\mathcal{X}_s) = \mathcal{X}_s/\mathcal{X}_s^o$ of connected components of $\mathcal{X}_s$. To see this, note that the obvious morphism of sheaves $\mathcal{F} \to \pi_0(\mathcal{X}_s)$ is surjective (by smoothness of $\mathcal{X}$) and that its kernel is precisely $\mathcal{F}^o$.

Proposition 5.2. *Let G be a semi-abelian K-variety, with Néron lft-model $\mathcal{G}$. Let K' be a finite separable extension of K, and denote by $\mathcal{G}'$ the Néron lft-model of $G' = G \times_K K'$. Using the notation from Section 2.2, there exists a canonical isomorphism*

$$h^* \Phi_{j_*(h_K)_* \mathscr{F}_{G'}} \cong \Phi_{j'_* \mathscr{F}_{G'}}$$

and these abelian sheaves on $(\mathrm{Spf}\, R')_{sm}$ are canonically isomorphic to the constant sheaf associated to the group $\Phi_{G'}$.

Proof. The sheaf $j'_*\mathscr{F}_{G'}$ is represented by the smooth formal R'-scheme $\widehat{\mathcal{G}'}$, so that $\Phi_{j'_*\mathscr{F}_{G'}}$ is canonically isomorphic to the constant sheaf $\Phi_{G'}$.

Since $(\mathcal{G}')^o$ is quasi-projective [6, 6.4.1], the Weil restriction

$$\prod_{R'/R} \mathcal{G}'$$

of $\mathcal{G}'$ with respect to $R \to R'$ is representable by a separated smooth group R-scheme $\mathcal{W}$ [6, 7.6.4+5]. Since Weil restriction commutes with base change, the generic fiber $W := \mathcal{W}_K$ is the Weil restriction of G' to K. It is obvious that $\mathcal{W}$ is a Néron lft-model for W.

By [5, 1.19], there is a canonical isomorphism

$$\mathscr{F}_W \cong (h_K)_*\mathscr{F}_{G'}$$

By the same arguments as above, the abelian sheaf

$$h^*\Phi_{j_*(h_K)_*\mathscr{F}_{G'}}$$

on $(\mathrm{Spf}\, R')_{sm}$ is canonically isomorphic to the constant sheaf Φ_W. Hence, it suffices to construct a canonical isomorphism between Φ_W and $\Phi_{G'}$.

For this, we can copy the last part of the proof of Theorem 1 in [11], where the authors construct a smooth surjective morphism of algebraic k'-groups

$$\mathcal{W}_s \times_k k' \to \mathcal{G}'_s$$

with connected kernel (in [11], it is assumed that G is an abelian variety, but the proof is also valid for semi-abelian varieties). $\qquad\square$

Corollary 5.3. *We keep the notations of Proposition 5.2. The natural morphism of group R'-schemes*

$$\mathcal{G} \times_R R' \to \mathcal{G}'$$

induces a morphism of abelian groups

$$\alpha : \Phi_G \to \Phi_{G'}$$

whose kernel is killed by $d = [K' : K]$.

Proof. We consider the sequence

$$\mathscr{F}_G \xrightarrow{\tau} (h_K)_*(h_K)^*\mathscr{F}_G \xrightarrow{tr} \mathscr{F}_G$$

where τ is the tautological morphism, and tr the trace map. We know that $tr \circ \tau$ is multiplication by d. Applying the functor j_* and passing to component groups, we obtain a sequence

$$\Phi_G \longrightarrow \Phi_{j_*(h_K)_*(h_K)^*\mathscr{F}_G} \longrightarrow \Phi_G$$

of abelian sheaves on $(\mathrm{Spf}\, R)_{sm}$. Applying h^*, and using Proposition 5.2, this yields a sequence of constant groups

$$\Phi_G \xrightarrow{\alpha} \Phi_{G'} \xrightarrow{\beta} \Phi_G$$

such that $\beta \circ \alpha$ is multiplication by d. The result follows. $\qquad\square$

In the case where G is an abelian variety, Corollary 5.3 is equivalent to Theorem 1 in [11].

Corollary 5.4. *Let G be a semi-abelian K-variety. Assume that G acquires good reduction over a finite separable field extension K' of K. Then the torsion part of Φ_G is killed by $d = [K' : K]$.*

Proof. We put $G' = G \times_K K'$. By Corollary 5.3, it suffices to show that $\Phi_{G'}$ has no torsion. Denote by

$$0 \to G'_{tor} \to G' \to G'_{ab} \to 0$$

the Chevalley decomposition of G'. Then G'_{tor} is a split torus, and G'_{ab} has good reduction. It follows from [8, 4.11] that the sequence

$$0 \to \Phi_{G'_{tor}} \to \Phi_{G'} \to \Phi_{G'_{ab}} \to 0$$

is exact. But $\Phi_{G'_{ab}} = 0$, and $\Phi_{G'_{tor}}$ is $\mathbb{Z}$-free. $\qquad\square$

Proposition 5.5. *Let G be a semi-abelian variety, and let L be a finite separable extension of K such that G acquires good reduction over L. Let K' be a finite separable extension of K such that $d = [K' : K]$ is prime to $e = [L : K]$. Put $G' = G \times_K K'$, and consider the morphism*

$$\alpha : \Phi_G \to \Phi_{G'}$$

from Corollary 5.3. Then the following hold:

(1) *the morphism α is injective*
(2) *if L is a tame extension of K, and $t(G) = 0$, then α is an isomorphism*
(3) *if G is a torus and $t(G) = 0$, then α is an isomorphism.*

Proof. (1) By Corollary 5.4, the torsion part of Φ_G is killed by e. Since the kernel of α is killed by d, by Corollary 5.3, and d is prime to e, the kernel of α must be trivial.

(2) Since $t(G) = t(G') = 0$ by Corollary 4.2, we know by Proposition 3.12 that G and G' do not admit a subgroup of type $\mathbb{G}_{m,k}$, so that the groups Φ_G and $\Phi_{G'}$ are finite [6, 10.2.1]. By (1), it suffices to show that Φ_G and $\Phi_{G'}$ have the same cardinality.

Since Φ_G and $\Phi_{G'}$ are killed by $e = [L : K]$, and L is a tame extension of K, the values $\phi(G)$ and $\phi(G')$ are prime to p. Therefore, it is enough to prove that $\phi(G)_q = \phi(G')_q$ for each prime $q \neq p$, where $\phi(G)_q$ denotes the q-primary part of $\phi(G)$.

If we put $I' = Gal(K^s/K')$, then

$$\begin{aligned} \phi(G)_q &= |H^1(I, T_q G)_{tors}| \\ \phi(G')_q &= |H^1(I', T_q G)_{tors}| \end{aligned}$$

by Proposition 4.4. If we put $I'' = Gal(K^s/L)$, then I'' acts trivially on $T_q G$, because G has potential good reduction (Proposition 4.3). Since $T_q G$ is torsion-free, the inflation morphisms

$$\begin{aligned} H^1(Gal(L/K), T_q G) &\to H^1(I, T_q G) \\ H^1(Gal(K'L/K'), T_q G) &\to H^1(I', T_q G) \end{aligned}$$

are isomorphisms [18, VII.6.Prop. 4]. Since $[L : K]$ is prime to $[K' : K]$, the restriction morphism

$$res : Gal(K'L/K') \to Gal(L/K)$$

is an isomorphism. It follows that

$$H^1(I, T_q G) \cong H^1(I', T_q G)$$

(3) As in (2), it suffices to show that $\phi(G) = \phi(G')$. Denote by $X(G)$ the character module of G. It follows from [4, 7.2.2] that

$$\begin{aligned} \phi(G) &= |H^1(Gal(L/K), X(G))| \\ \phi(G') &= |H^1(Gal(K'L/K'), X(G))| \end{aligned}$$

Since the restriction morphism $res : Gal(K'L/K') \to Gal(L/K)$ is an isomorphism, we find

$$\phi(G) = \phi(G')$$

$\square$

Corollary 5.6. *Let G be a semi-abelian variety, and let L be a finite separable extension of K such that G acquires good reduction over L. Let K' be a finite separable extension of K such that $d = [K' : K]$ is prime to $e = [L : K]$. Put $G' = G \times_K K'$, and consider the morphism*

$$\alpha : \Phi_G \to \Phi_{G'}$$

from Corollary 5.3. Then the following hold:

(1) *if L is a tame extension of K, then*

$$|coker(\alpha)| = d^{t(G)}$$

(2) *if G is a torus, then*

$$|coker(\alpha)| = d^{t(G)}$$

Proof. Denote by T_{sp} the maximal split subtorus of G. By the proof of [8, 4.11], the morphism α fits into a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_{T_{sp}} & \longrightarrow & \Phi_G & \longrightarrow & \Phi_{G/T_{sp}} & \longrightarrow & 0 \\ & & \beta \downarrow & & \alpha \downarrow & & \downarrow \gamma & & \\ 0 & \longrightarrow & \Phi_{(T_{sp})'} & \longrightarrow & \Phi_{G'} & \longrightarrow & \Phi_{G'/(T_{sp})'} & \longrightarrow & 0 \end{array}$$

where $(\cdot)'$ denotes base change to K'. Under the hypotheses of (1) or (2), the morphism γ is an isomorphism, by Propositions 3.12 and 5.5. By Lemma 5.1, we get

$$|coker(\alpha)| = |coker(\beta)|$$

It follows from the description of the Néron model of $\mathbb{G}_{m,K}$ in [6, 10.1.5] that

$$|coker(\beta)| = d^{\dim(T_{sp})} = d^{t(G)}$$

where the equality $\dim(T_{sp}) = t(G)$ follows from Proposition 3.12. $\square$

The key result of the present paper is the following theorem.

Theorem 5.7. *Let A be an abelian K-variety, with Néron model $\mathcal{A}$, and denote by $t(A)$ the reductive rank of $\mathcal{A}_s^o$. We denote by $e \in \mathbb{Z}_{>0}$ the degree of the minimal extension of K where A acquires semi-abelian reduction.*

Let K' be a finite separable extension of K, and put $d = [K' : K]$. If d is prime to e, then the natural map

$$\Phi_A \to \Phi_{A \times_K K'}$$

is injective. If A is tamely ramified, or A has potential purely multiplicative reduction, then

$$\phi(A \times_K K') = d^{t(A)}\phi(A)$$

Proof. We fix an embedding of K' in K^s. We'll use the notation from Section 2.2. The short exact sequence of rigid K-groups

$$0 \to M^{an} \to E^{an} \to A^{an} \to 0$$

gives rise to an exact sequence

$$0 \to \mathscr{F}_M \to \mathscr{F}_E \to \mathscr{F}_A \to 0$$

of abelian sheaves on $(\operatorname{Sp} K)_{sm}$ (right exactness of this sequence follows from the smoothness of $E^{an} \to A^{an}$). By Lemma 2.1 we find a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}_M & \longrightarrow & \mathscr{F}_E & \longrightarrow & \mathscr{F}_A & \longrightarrow & 0 \\
& & \tau \downarrow & & \tau \downarrow & & \downarrow \tau & & \\
0 & \longrightarrow & (h_K)_* \mathscr{F}_{M'} & \longrightarrow & (h_K)_* \mathscr{F}_{E'} & \longrightarrow & (h_K)_* \mathscr{F}_{A'} & \longrightarrow & 0 \\
& & tr \downarrow & & tr \downarrow & & \downarrow tr & & \\
0 & \longrightarrow & \mathscr{F}_M & \longrightarrow & \mathscr{F}_E & \longrightarrow & \mathscr{F}_A & \longrightarrow & 0
\end{array}
$$

where τ is the tautological morphism, and tr is the trace map (Section 2.3). We know that $tr \circ \tau$ is multiplication by d.

We put $I' = Gal(K^s/K')$. Applying the functor j_*, and using [8, 4.4], we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^I & \longrightarrow & j_* \mathscr{F}_E & \longrightarrow & j_* \mathscr{F}_A & \xrightarrow{\gamma_1} & H^1(I, M) \\
& & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \downarrow \alpha_3 & & \downarrow \alpha_4 \\
0 & \longrightarrow & M^{I'} & \longrightarrow & j_*(h_K)_* \mathscr{F}_{E'} & \longrightarrow & j_*(h_K)_* \mathscr{F}_{A'} & \xrightarrow{\gamma_2} & H^1(I', M) \\
& & \beta_1 \downarrow & & \beta_2 \downarrow & & \downarrow \beta_3 & & \downarrow \beta_4 \\
0 & \longrightarrow & M^I & \longrightarrow & j_* \mathscr{F}_E & \longrightarrow & j_* \mathscr{F}_A & \longrightarrow & H^1(I, M)
\end{array}
$$

where $\beta_i \circ \alpha_i$ is multiplication by d, for $i = 1, \ldots, 4$. Here we view M as a discrete I-module, and the objects in the second and fifth columns as constant abelian sheaves on $(\operatorname{Spf} R)_{sm}$.

Claim 1: α_1 is an isomorphism. Since A acquires semi-abelian reduction over L, the I-action on M factors through $Gal(L/K)$. Since L and K' are linearly disjoint over K, the restriction morphism

$$res : Gal(K'L/K') \to Gal(L/K)$$

is an isomorphism, and $M^I = M^{I'}$.

Claim 2: α_4 and β_4 are isomorphisms. Consider the commutative diagram

$$
\begin{array}{ccc}
H^1(I, M) & \xrightarrow{\alpha_4} & H^1(I', M) \\
inf \uparrow & & \uparrow inf' \\
H^1(Gal(L/K), M) & \xrightarrow{\sim} & H^1(Gal(K'L/K'), M)
\end{array}
$$

where the vertical arrows are the inflation morphisms, and the lower horizontal isomorphism is induced by the isomorphism

$$res : Gal(K'L/K') \to Gal(L/K)$$

Since $Gal(K^s/L)$ acts trivially on M, and M is torsion-free, the morphisms inf and inf' are isomorphisms [18, VII.6.Prop. 4], so that α_4 is an isomorphism, too. The isomorphism inf shows that $H^1(I, M)$ is killed by e. Since d is prime to e, the composition $\beta_4 \circ \alpha_4$, and hence the morphism β_4, are isomorphisms.

Now we pass to component groups. Using [8, 4.12] and Proposition 5.2, we find a commutative diagram of constant groups, with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^I & \longrightarrow & \Phi_E & \longrightarrow & \Phi_A & \xrightarrow{\widetilde{\gamma}_1} & H^1(I, M) \\
 & & \alpha_1 \downarrow \wr & & \widetilde{\alpha}_2 \downarrow & & \widetilde{\alpha}_3 \downarrow & & \wr \downarrow \alpha_4 \\
0 & \longrightarrow & M^{I'} & \longrightarrow & \Phi_{E'} & \longrightarrow & \Phi_{A'} & \xrightarrow{\widetilde{\gamma}_2} & H^1(I', M) \\
 & & & & & & \widetilde{\beta}_3 \downarrow & & \wr \downarrow \beta_4 \\
 & & & & & & \Phi_A & \xrightarrow{\widetilde{\gamma}_1} & H^1(I, M)
\end{array}
$$

Since $\widetilde{\alpha}_2$ is injective by Corollary 5.6, a diagram chase shows that $\widetilde{\alpha}_3$ is injective. Hence, from now on, we may assume that A is tamely ramified (so that E is tamely ramified) or A has potential purely multiplicative reduction (so that E is a torus).

Claim 3: the isomorphism α_4 identifies the images of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. We denote by D_A and D'_A the subgroups of Φ_A, resp. $\Phi_{A'}$, consisting of the elements that are killed by a power of d. Since $H^1(I, M)$ and $H^1(I', M)$ are killed by e, the morphisms $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are trivial on D_A, resp. D'_A, and we obtain a commutative diagram

$$
\begin{array}{ccc}
\Phi_A/D_A & \xrightarrow{\overline{\gamma}_1} & H^1(I, M) \\
\overline{\alpha}_3 \downarrow & & \wr \downarrow \alpha_4 \\
\Phi_{A'}/D_{A'} & \xrightarrow{\overline{\gamma}_2} & H^1(I', M) \\
\overline{\beta}_3 \downarrow & & \wr \downarrow \beta_4 \\
\Phi_A/D_A & \xrightarrow{\overline{\gamma}_1} & H^1(I, M)
\end{array}
$$

Since $\overline{\beta}_3 \circ \overline{\alpha}_3$ is multiplication by d, it is injective, and hence surjective because Φ_A is finite. A diagram chase shows that α_4 identifies the images of $\overline{\gamma}_1$ and $\overline{\gamma}_2$.

By Lemma 5.1, we know that $coker(\widetilde{\alpha}_2) \cong coker(\widetilde{\alpha}_3)$. Therefore, it suffices to prove that

$$|coker(\widetilde{\alpha}_2)| = d^{t(A)}$$

This follows from Corollary 5.6. $\qquad\square$

Remarks. (1) Theorem 5.7 remains valid if we replace the general assumption that K is complete, by the assumption that K is strictly Henselian. It suffices to note that $\phi(A)$ is invariant under base change to the completion $\widehat{K}$ [6, 7.2.1].

(2) The second part of Theorem 5.7 does not hold for arbitrary wildly ramified abelian K-varieties. A counterexample is provided in Example 5.8 below. Note however that Corollary 5.6(3), which is the analog of Theorem 5.7 for tori, does not require any tameness assumption.

(3) It seems plausible that Theorem 5.7 holds also for tamely ramified semi-abelian K-varieties G, in the following form: the map

$$\Phi_G \to \Phi_{G \times_K K'}$$

is injective, and its cokernel has cardinality $d^{t(G)}$.

(4) From the obvious sequence of maps

$$T^{an}_{sp} \to T^{an} \to E^{an} \to A^{an}$$

one derives a sequence of component groups

$$\Phi_{T_{sp}} \to \Phi_T \to \Phi_E \to \Phi_A$$

Taking images in Φ_A one obtains, in a canonical way, a filtration

$$0 = \Sigma_4 \subset \Sigma_3 \subset \Sigma_2 \subset \Sigma_1 \subset \Phi_A$$

which was introduced in [8, §5].

Let us assume that A acquires semi-abelian reduction over a tame extension of K of degree e. Consider a tame extension K'/K of degree d, with d prime to e. If we put $A' = A \times_K K'$ and denote by $\Sigma'_\bullet$ the filtration on $\Phi_{A'}$, we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 = \Sigma_4 & \longrightarrow & \Sigma_3 & \longrightarrow & \Sigma_2 & \longrightarrow & \Sigma_1 & \longrightarrow & \Sigma_0 = \Phi_A \\
& & \gamma_3 \downarrow & & \gamma_2 \downarrow & & \gamma_1 \downarrow & & \downarrow \\
0 = \Sigma'_4 & \longrightarrow & \Sigma'_3 & \longrightarrow & \Sigma'_2 & \longrightarrow & \Sigma'_1 & \longrightarrow & \Sigma'_0 = \Phi_{A'}
\end{array}
$$

where all maps are injections. For all $0 \le i < j \le 4$, we denote by

$$\gamma_{i,j} : \Sigma_i/\Sigma_j \to \Sigma'_i/\Sigma'_j$$

the map induced by γ_i. Then one can show that $|coker(\gamma_3)| = d^{t(A)}$, and that $\gamma_{i,j}$ is an isomorphism whenever $0 \le i < j \le 3$. □

Example 5.8. *In this example we take R to be the Witt vectors $W(k)$, with k an algebraically closed field of characteristic 2. Let C be the elliptic K-curve with Weierstrass equation*

$$y^2 = x^3 + 2$$

Recall that 2 is a uniformizing parameter for R.

It is easily computed, using Tate's algorithm, that C has reduction type II over R. Consider $L = K(2^{1/2})$, which is a wild Kummer extension of K with $[L : K] = 2$. Then $C \times_K L$ has good reduction.

On the other hand, let $K(3) := K(2^{1/3})$, which is a tame extension of K with $[K(3) : K] = 3$. One checks that $C \times_K K(3)$ has reduction type I_0^* over $R(3)$. In particular,

$$0 = \Phi_C \neq \Phi_{C \times_K K(3)} = (\mathbb{Z}/2\mathbb{Z})^2$$

We conclude that the second statement in Theorem 5.7 does not hold for C. Nevertheless, we will see in Lemma 6.7 that Theorem 5.7 holds for all elliptic curves over K if we replace e by another invariant of the curve.

Example 5.9. *Let C be a tamely ramified elliptic curve over K with additive reduction. In view of Theorem 5.7, one might be tempted to think that C and $C \times_K K'$ have the same reduction type if K' is a finite extension of K such that $[K' : K]$ is prime to the degree of the minimal extension of K where C acquires semi-abelian reduction. However, this is not the case.*

Consider, for instance, the elliptic K-curve C with Weierstrass equation

$$y^2 = x^3 + \pi^4$$

where π is a uniformizer in R. We assume that $p > 3$. In particular, C is tamely ramified. The minimal extension of K where C acquires semi-abelian reduction is $K(3)$.

Using Tate's algorithm, one computes that C has reduction type IV^* over R, and that for each $n \in \mathbb{N}'$ such that $n \equiv 2$ modulo 6, the elliptic curve $C \times_K K(n)$ has reduction type IV over $R(n)$.

6. RATIONALITY OF THE NÉRON COMPONENT SERIES

6.1. **Rationality of the component series for tamely ramified abelian varieties.**

Lemma 6.1. *Let $P(t) = \sum_{i>0} p_i t^i$ and $Q(t) = \sum_{i>0} q_i t^i$ be non-zero power series in $\mathbb{Z}[[t]]$ such that $p_i, q_i \geq 0$ for all $i > 0$. Assume that $P(t)$ and $Q(t)$ converge on the open complex unit disc D, and that they have a pole of order m_P, resp. m_Q, at $t = 1$. Then $P(t) + Q(t)$ has a pole of order $\max\{m_P, m_Q\}$ at $t = 1$.*

Proof. We may assume that $m_P = m_Q =: m$. It suffices to show that the residues of $P(t)$ and $Q(t)$ at $t = 1$ have the same sign. We denote these residues by ρ_P and ρ_Q, respectively. Let $(t_n)_{n \geq 0}$ be a series in $]0, 1[$ that converges to 1. Since the coefficients of $P(t)$ are positive, $P(t)$ takes positive values on $]0, 1[$, so that

$$(-1)^m \rho_P = \lim_{n \to \infty} (1 - t_n)^m P(t_n) > 0$$

Likewise, $(-1)^m \rho_Q > 0$. □

Lemma 6.2. *For each element a of $\mathbb{N}$, the series*

$$\psi_a(T) = \sum_{d>0} d^a T^d$$

belongs to

$$\mathbb{Z}\left[T, \frac{1}{T-1}\right]$$

It has degree zero if $a = 0$ and degree < 0 else. It has a pole of order $a+1$ at $T = 1$, whose residue equals $(-1)^{a+1} a!$.

Proof. It suffices to show that $\psi_a(T)$ is a rational function in T of the form

$$\frac{R_a(T)}{(T-1)^{a+1}}$$

with $R_a(T) \in \mathbb{Z}[T]$, $\deg R_a(T) \leq \max\{a, 1\}$ and $R_a(1) = (-1)^{a+1} a!$. We proceed by induction on a. For $a = 0$ the result is clear, so assume that $a > 0$ and that the assertion holds for all $\psi_{a'}(T)$ with $0 \leq a' < a$.

Denoting by ∂_T the derivation w.r.t. the variable T, we have

$$\psi_a(T) = T\sum_{d>0}(d^a T^{d-1}) = T\partial_T\left(\sum_{d>0} d^{a-1}T^d\right) = T\partial_T\psi_{a-1}(T)$$
$$= \frac{T((T-1)\partial_T R_{a-1}(T) - a \cdot R_{a-1}(T))}{(T-1)^{a+1}}$$

so the result follows from the induction hypothesis and the fact that $R_0(T) = T$. $\square$

Definition 6.3. *We define the tame potential toric rank $t_{tame}(A)$ of an abelian K-variety A by*

$$t_{tame}(A) = \max\{t(A\times_K K')\,|\,K' \text{ a finite tame extension of } K\}$$

If A is tamely ramified, then $t_{tame}(A) = t_{pot}(A)$.

Lemma 6.4. *Let A be an abelian K-variety, and denote by e the degree of the minimal extension of K where A acquires semi-abelian reduction. If we denote by e' the prime-to-p part of e, then*

$$t_{tame}(A) = t(A\times_K K(e'))$$

Proof. By Proposition 3.9, we may assume that $e' = 1$, and it suffices to show that

$$t(A) = t(A\times_K K')$$

for every finite tame extension K' of K. This follows from Corollary 4.2. $\square$

Theorem 6.5. *Let A be an abelian K-variety. Assume that A is tamely ramified, or that A has potential purely multiplicative reduction. The Néron component series*

$$S_\phi(A;T) = \sum_{d\in\mathbb{N}'} \phi(A\times_K K(d))T^d$$

belongs to

$$\mathscr{Z} := \mathbb{Z}\left[T, \frac{1}{T^j - 1}\right]_{j\in\mathbb{Z}_{>0}}$$

It has degree zero if $p = 1$ and A has potential good reduction, and degree < 0 in all other cases. It has a pole at $T = 1$ of order $t_{tame}(A) + 1$.

Proof. For notational convenience, we put $A(d) = A\times_K K(d)$ for every element d of $\mathbb{N}'$. We denote by e the degree of the minimal extension of K where A acquires semi-abelian reduction, and by $\mathscr{D}_e$ the set of divisors of e that belong to $\mathbb{N}'$. We introduce the series

$$S'_\phi(A;T) = \sum_{d\in\mathbb{N}',\,gcd(d,e)=1} \phi(A(d))T^d$$

Then we can write

$$S_\phi(A;T) \;=\; \sum_{a\in\mathscr{D}_e}\sum_{d\in\mathbb{N}',\,gcd(d,e)=a} \phi(A(d))T^d$$
$$=\; \sum_{a\in\mathscr{D}_e} S'_\phi(A(a);T^a)$$

because the degree of the minimal extension of $K(a)$ where $A(a)$ acquires semi-abelian reduction is equal to e/a, for each a in $\mathscr{D}_e$.

By Lemma 6.4, we have

$$t_{tame}(A) = \max\{t(A(a)) \,|\, a \in \mathscr{D}_e\}$$

In view of Lemma 6.1, it suffices to prove the following claims:

(1) for each $a \in \mathscr{D}_e$, the series $S'_\phi(A(a); T^a)$ belongs to $\mathscr{Z}$. It has a pole at $T = 1$ of order $t(A(a)) + 1$.
(2) The degree of $S'_\phi(A(a); T^a)$ is
   - zero if $p = 1$, $a = e$, and $t(A(e)) = 0$,
   - strictly negative in all other cases.

First, we prove (1). It suffices to consider the case $a = 1$. We denote by $\mathscr{P}_e$ the set of elements in $\{1, \ldots, e\}$ that are prime to e. If $p \nmid e$, then for each $b \in \mathscr{P}_e$, we denote by n_b the smallest element of $b + \mathbb{N}e$ such that $n_b \notin \mathbb{N}'$. If $p | e$, we put $n_b = 0$ for each $b \in \mathscr{P}_e$. Note that, in any case, $n_b < pe$. We put $\varepsilon_k = 0$ if $p|e$, and $\varepsilon_k = 1$ else.

By Theorem 5.7, we have

$$S'_\phi(A; T) \;=\; \sum_{d \in \mathbb{N}',\, gcd(d,e)=1} d^{t(A)} \phi(A) T^d$$

$$= \; \phi(A) \cdot \sum_{b \in \mathscr{P}_e} \left(\sum_{q \in \mathbb{N}} (qe + b)^{t(A)} T^{qe+b} - \varepsilon_k \sum_{r \in \mathbb{N}} (n_b + epr)^{t(A)} T^{n_b + epr} \right)$$

By Lemma 6.2, the series

$$(6.1) \qquad\qquad \sum_{q \in \mathbb{N}} (qe + b)^{t(A)} T^{qe+b}$$

belongs to $\mathscr{Z}$, for each $b \in \mathscr{P}_e$. It has a pole of order $t(A) + 1$ at $T = 1$, and the residue of this pole equals

$$(-1)^{t(A)+1} e^{-1} (t(A)!)$$

Likewise, the series

$$(6.2) \qquad\qquad \sum_{r \in \mathbb{N}} (n_b + epr)^{t(A)} T^{n_b + epr}$$

belongs to $\mathscr{Z}$. It has a pole of order $t(A) + 1$ at $T = 1$, and the residue equals

$$(-1)^{t(A)+1} (ep)^{-1} (t(A)!)$$

It follows that $S'_\phi(A; T)$ belongs to $\mathscr{Z}$, and that it has a pole of order $t(A) + 1$ at $T = 1$.

Now we prove claim (2). If $t(A) > 0$, then it follows easily from Lemma 6.2 that the series (6.1) and (6.2) have degree < 0, so that $S'_\phi(A(a); T^a)$ has degree < 0 if $t(A(a)) > 0$.

If $t(A) = 0$, then we find

$$S'_\phi(A; T) = \phi(A) \cdot \sum_{b \in \mathscr{P}_e} \left(\frac{T^b}{1 - T^e} - \varepsilon_k \frac{T^{n_b}}{1 - T^{ep}} \right)$$

This rational function has degree ≤ 0, and its degree is equal to zero iff $e = 1$ and $p = 1$. Replacing A by the abelian varieties $A(a)$, for $a \in \mathscr{D}_e$, we obtain the required result. $\qquad\square$

We expect that Theorem 6.5 holds for all abelian K-varieties. In the following section, we'll prove that this is true for elliptic curves.

6.2. **Rationality of the component series for wildly ramified elliptic curves.** Assume that k is algebraically closed. Let C be a smooth, proper, geometrically connected K-curve of genus $g(C) > 0$, and let $\mathcal{C}/R$ be the minimal sncd-model of C (i.e., the minimal regular model with strict normal crossings [15, 10.1.8]). We assume that either $g(C) > 1$, or C is an elliptic curve.

We call an irreducible component E of $\mathcal{C}_s$ *principal* if $g(E) > 0$ or E intersects the other components in at least three distinct points. Let $e(C)$ be the least common multiple of the multiplicities of the principal components of $\mathcal{C}_s$, and let $e(C)'$ be the prime-to-p part of $e(C)$.

If C is tamely ramified (i.e., the wild inertia acts trivially on the ℓ-adic cohomology of C), then $e(C) = e(C)'$ is equal to the degree e of the minimal extension of K where C acquires semi-stable reduction, by [12, 7.5] (to be precise, in [12] it is assumed that $g(C) > 1$, but the proof applies to elliptic curves as well). On the other hand, if C is the (wildly ramified) elliptic curve from Example 5.8, then $e(C) = 6$ while $e = 2$, so that not even their prime-to-p parts coincide.

Lemma 6.6. *Assume that k is algebraically closed. Let C be an elliptic curve over K, and let a be a divisor of $e(C)'$. Then $e(C \times_K K(a))' = e(C)'/a$.*

Proof. If C is tamely ramified, this follows from the equality $e = e(C)$. Hence, we may assume that C is wildly ramified. Considering the Kodaira classification and applying Saito's criterion for wild ramification [17, 3.11], $e(C)' > 1$ only occurs when p is either 2 or 3 and the reduction type of C is either II or II^*. We will give a detailed argument when $p = 2$ and C has reduction type II, the remaining cases follow in a similar fashion. In this case, we have $e(C)' = 3$. For $a = 1$ there is nothing to prove, so we may assume that $a = 3$.

We will use the computations and results from [12]. It is a slight problem that there exists a pair of intersecting components of $\mathcal{C}_s$ that both have multiplicities divisible by p, since locally at such intersection points, the methods of [12] don't apply. However, because of the very limited possibilities of degeneration types for elliptic curves, we can get around this with some ad hoc arguments.

The special fiber $\mathcal{C}_s$ is of the form

$$\mathcal{C}_s = E_1 + 2E_2 + 3E_3 + 6E_4$$

where E_4 meets each other component transversally in a unique point, and the other components are pairwise disjoint. We denote by $\mathcal{D}$ the normalization of $\mathcal{C} \times_R R(3)$. It follows from [12, 2.1+2.9+6.3] that

$$\mathcal{D}_s = F_1 + 2F_2 + F_3^1 + F_3^2 + F_3^3 + 2F_4$$

where F_i dominates E_i, for $i = 1, 2, 4$, and F_3^j dominates E_3, for $j = 1, 2, 3$. Moreover, F_4 intersects F_3^j and F_1 transversally at a unique point, and $F_2 \cap F_4 \neq \emptyset$. It follows from [12, 4.3] that $\mathcal{D}$ is regular outside of $F_2 \cap F_4$, and from [12, 2.2+2.9] that all the components of $\mathcal{D}_s$ are smooth, except possibly for F_2 and F_4 at the points where they intersect.

Let

$$\rho : \mathcal{C}(3) \to \mathcal{D}$$

be the minimal sncd-desingularization. It follows from what we've said above that ρ is an isomorphism above $\mathcal{D} \setminus \{F_2 \cap F_4\}$. Let

$$\tau : \mathcal{C}(3) \to \mathcal{C}(3)_{min}$$

be the canonical morphism to the minimal sncd-model $\mathcal{C}(3)_{min}$ of $C \times_K K(3)$. It is obvious that τ is an open immersion when restricted to $\rho^{-1}(\mathcal{D} \setminus \{F_2 \cap F_4\})$. It follows that the special fiber of $\mathcal{C}(3)_{min}$ contains a component with multiplicity 2, meeting four reduced components each in a unique point. The only possibility is then reduction type I_0^*.

These are the results in the other cases:

- if $p = 2$ and C has type II^*, then $e(C)' = 3$ and $C \times_K K(3)$ has type I_0^*,
- if $p = 3$ and C has type II, then $e(C)' = 2$ and $C \times_K K(2)$ has type IV,
- if $p = 3$ and C has type II^*, then $e(C)' = 2$ and $C \times_K K(2)$ has type IV^*.

$\square$

Lemma 6.7. *Assume that k is algebraically closed. Let C be an elliptic curve that does not have multiplicative reduction. For every finite tame extension K'/K whose degree is prime to $e(C)$, we have*

$$\phi(C \times_K K') = \phi(C)$$

Proof. If C is tamely ramified, this follows from Theorem 5.7. We give an alternative proof that is valid also in the wild case. We may assume that C has additive reduction. Looking at the Kodaira reduction table, one sees that the special fiber of the minimal *sncd*-model $\mathcal{C}$ of C contains a principal component, that all the principal components of $\mathcal{C}_s$ have the same multiplicity m, and that this multiplicity m determines $\phi(C)$ (the principal component is unique except in the case where C has reduction type I_n^* with $n > 0$).

Explicitly, we have

$$m = \begin{cases} 1 & \text{if C has type} & I_0 \\ 2 & & I_n^*,\ n \geq 0 \\ 6 & & II \text{ or } II^* \\ 4 & & III \text{ or } III^* \\ 3 & & IV \text{ or } IV^* \end{cases}$$

Reasoning as in the proof of Lemma 6.6, one sees that each principal component of $\mathcal{C}_s$ gives rise to a principal component with the same multiplicity in the minimal *sncd*-model of $C \times_K K'$. Hence,

$$\phi(C \times_K K') = \phi(C)$$

$\square$

Proposition 6.8. *Assume that k is algebraically closed. Let C be an elliptic curve over K. The component series*

$$S_\phi(C; T) = \sum_{d \in \mathbb{N}'} \phi(C \times_K K(d)) T^d$$

belongs to

$$\mathscr{L} := \mathbb{Z}\left[\frac{T^j}{1 - T^j}\right]_{j \in \mathbb{Z}_{>0}}$$

It has a pole at $T = 1$ of order $t_{tame}(C) + 1$. It has degree zero if $p = 1$ and C has potential good reduction, and degree < 0 in all other cases.

Proof. If C is tamely ramified, then this follows from Theorem 6.5. Hence, we may assume that C is wildly ramified. In this case, $p > 1$ and $t_{tame}(C) = 0$. For notational convenience, we put $C(d) = C \times_K K(d)$ for each $d \in \mathbb{N}'$.

We can write

$$S_\phi(C; T) = \sum_{a | e(C)'} \sum_{d \in \mathbb{N}', gcd(d, e(C)') = a} \phi(C(d)) T^d$$

By Lemmas 6.6 and 6.7, this expression equals

$$\sum_{a | e(C)'} \left(\phi(C \times_K K(a)) \cdot \sum_{d \in \mathbb{N}', gcd(d, e(C(a))') = 1} T^{ad} \right)$$

Direct computation shows that this series belongs to $\mathscr{L}$, that it has a pole of order one at $T = 1$, and that it has strictly negative degree. $\qquad\square$

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